Differentiating the above expressions by t, replacing $lE(r_1, t)$ and $lE(r_2, t)$ by I_1R_1 and I_2R_2, respectively, and using the results in Eq. (A1) will yield Eq. (1) of the text.

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On the Repulsion of Slow Neutrons by Attractive Potentials*

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A naive calculation of the scattering of slow neutrons by an attractive square-well potential gives a distribution of scattering amplitudes that agrees rather well with available measured amplitudes for thermal neutrons.

I. INTRODUCTION

To a slow neutron, all nuclei are attractive potential wells from 10 to 60 MeV deep and a few times 10^{-13} cm in diameter. Nevertheless, most elements, when made into mirrors, reflect (i.e., repel) slow neutrons.[1] Why?

Let us state the question more precisely. Since slow neutrons (velocities of a few meters per second) are totally externally reflected at all angles of incidence by most materials, the index of refraction n of slow neutrons in most materials is less than unity. That such an index of refraction is consistent with nuclear repulsion is verified[2] by calculation from nuclear theory, which gives

$$n = 1 - (\lambda^2 N / 2\pi) b, \qquad (1)$$

where λ is the neutron wavelength ($\sim 10^{-8}$ cm for thermal neutrons), N is the number of nuclei per unit volume, and the nuclear scattering amplitude b ($\sim 10^{-13}$ cm) is positive for a repulsive potential.[3] For a sufficiently weak attractive potential, b is negative, but b can have either sign for a strong attractive potential. Why then is the slow-neutron scattering amplitude "repulsive" for almost all nuclei?

The traditional simplest answer[4] to this question is that the incident neutron wave is strongly reflected by the sharp edge of the potential well which represents the interaction of the neutron with the nucleus. The kinetic energy of the neutron inside the well is much greater than the energy outside. The wavelength is therefore much shorter inside than outside. Except for a small range of nuclear diameters that correspond to an odd number of half-wavelengths, continuity of the wave function and its derivative then requires that the wave function be much smaller inside than outside, as illustrated in Fig. 1. In general, the neutron is almost excluded from the nuclear volume, just as it would be by a strong repulsive potential. This is in effect a repulsion from the

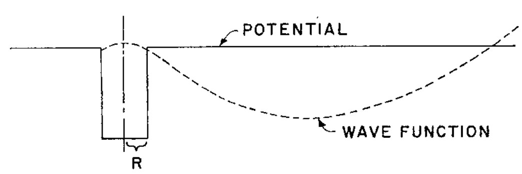

FIG. 1. The neutron wave function in the vicinity of an attractive potential of radius R.

material. The effect is a small one because the nuclear volume is small, so the index of refraction differs from unity by only a few parts in a million. The effect is so small that only those neutrons whose energy is less than 1 μeV can be totally reflected at normal incidence.

If the inside wave function has an antinode near the nuclear surface, as it will near a scattering resonance,[5] this argument fails. Then the index of refraction may be expected to have a wide range of values in a minority of the elements. Experimentally, there are about 20 such anomalous cases among approximately 130 known slow-neutron scattering amplitudes.[6] Can the simple theory be extended to give a simple semiquantitative account of the number of anomalous cases? In fact it can.

II. RANDOM-POTENTIAL MODEL

Represent the nucleus by the attractive potential

$$V(r) = -V_0 \quad \text{for} \quad r \leq R$$
$$= 0 \quad \text{for} \quad r > R, \tag{2}$$

where R is the nuclear radius. The wave function ψ of the neutron, with the convention used here,[3] is given by

$$\psi = \exp(ikx) - (b/r)\exp(ikr) \tag{3}$$

for $r > R$, where k is related to the neutron mass M and energy E through $k^2 = 2ME/\hbar^2$. For slow neutrons, only the s-wave part of ψ is involved in the scattering. That part is obtained from the exact solution of the Schrödinger equation as

$$\psi_s = A(\sin Kr)/Kr \qquad \text{for} \quad r \leq R$$
$$= (\sin kr)/kr - (b/r)\exp(ikr) \quad \text{for} \quad r > R, \tag{4}$$

where $K^2 = 2M(E + V_0)/\hbar^2$, and A and b must be adjusted to make ψ_s and $d\psi_s/dr$ continuous at $r = R$. The continuity condition gives

$$K \cot KR = \frac{\cos kR - ikb\exp(ikR)}{(\sin kR)/k - b\exp(ikR)}. \tag{5}$$

In the slow-neutron limit ($kR \to 0$), KR approaches

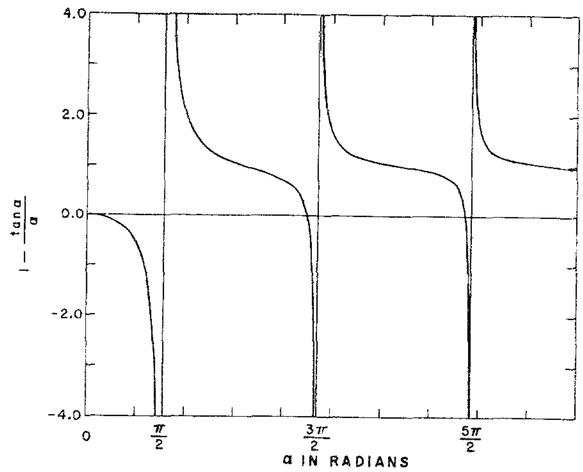

FIG. 2. The ratio of scattering amplitude to well radius, plotted as a function of $\alpha = (2MV_0)^{1/2}R/\hbar$.

α and Eq. (5) becomes

$$b/R = 1 - (1/\alpha)\tan\alpha, \tag{6}$$

where $\alpha^2 = 2MV_0R^2/\hbar^2$.

Equation (6) is plotted in Fig. 2 which, in appropriate units, is a graph of scattering amplitude vs square root of the well depth. For shallow wells, the scattering amplitude is negative (attractive) as expected. The infinities of the graph, which occur at the onset of bound states in the potential well, arise only in the limit $kR \to 0$. It is apparent from Fig. 2 that b/R is close to the "strong repulsive" value for a large fraction of the values of α. From Fig. 2, we can also construct a simple approximation to the distribution of the observed scattering amplitudes by assuming that the values of α are distributed uniformly at random over some reasonable range. The calculated distribution of scattering amplitudes will of course depend to some extent upon the range of α assumed.

In Fig. 3, the experimental distribution of scattering amplitudes is compared with the distribution calculated from Eq. (6). That comparison involves three ambiguities which appear not to becloud the main point. (1) The calculated distribution is presented for two choices of the range of α. In both cases, the lower limit is taken to be $\alpha = \pi/2$ because only two known nuclei have potential wells too shallow to bind a neutron. The

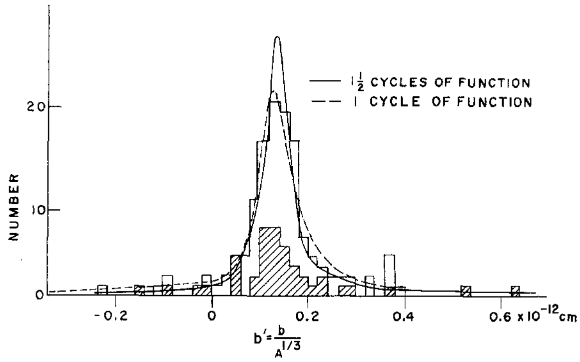

FIG. 3. The distribution of experimental scattering amplitudes of thermal neutrons. The cross-hatched histogram includes only amplitudes of definite spin states of single isotopes. The unshaded histogram includes all the data except five measured amplitudes: ^{36}A and ^{164}Dy because their (positive) values are too large to plot conveniently and ^{1}H, ^{4}He, and ^{113}Cd for reasons mentioned in the text. The smooth curves are distributions calculated from Eq. (6) on the basis of two assumed ranges of α, as explained in the text.

upper limits, $3\pi/2$ for the dashed curve and 2π for the solid curve were chosen to illustrate the extent of the ambiguity; other reasonable assumptions about the distribution of α give curves similar to those in Fig. 3. (2) The comparison requires an assignment of the nuclear radius R. For convenience, the experimental cases (represented by the unshaded histogram) are presented as a function of $b'=b/A^{1/3}$, where b is the measured scattering amplitude and A the atomic number. To represent the calculated ratios b/R on the same scale, we assumed that $R=R_0A^{1/3}$ and adjusted R_0 to make the maximum of the theoretical curves agree with that of the histogram. For both theoretical curves, we obtained $R_0=1.4F$, a large but not ridiculous figure. (3) Many of the published scattering amplitudes are averages over several isotopes or over two spin states. This average should narrow the experimental distribution relative to a completely analyzed distribution, but probably does not change the qualitative success of the model. The shaded part of the histogram, which includes only the unambiguous cases, appears to be similar in shape to the full histogram.

III. SOME SPECIAL CASES

If the scattering nucleus cannot bind a neutron, there is some question as to what cycle of Fig. 2 it belongs in. There are five such cases: ^{1}H (singlet state), ^{2}H (quartet), ^{3}H (both states) and ^{4}He. In the first there is clearly no bound state in the potential at all, and that case belongs in the first cycle. It has a relatively large negative scattering amplitude as would be expected since the lowest resonance in the n–^{1}H (singlet) system is only a little above the thermal-neutron energy. In the other cases, there are bound neutron levels but they are all full. This situation has been extensively studied by Swan,[7] who concludes that the filled levels act like empty levels as far as their effect on the sign of the scattering amplitude is concerned. The ^{2}H quartet state is known[8] to follow this prediction and ^{4}He probably does too.[9] However, there is a small chance that ^{4}He has a negative amplitude since that amplitude has not been measured directly. Hydrogen 3 is known[10] to have a positive average scattering amplitude (4.7 Fm) for the element; but since the two spin states have not been distinguished, there is again a weak possibility that one of them has a negative amplitude. A complication in the case of ^{3}H is that there exists a published[11] total cross section, $1.30\pm0.03b$, which is considerably smaller than $4\pi(4.7\times10^{-13})^2$.

ACKNOWLEDGMENTS

We are much indebted to S. L. Ruby for forcing us to attempt a simple answer to a simple question and to J.P. Schiffer for pointing out some complications.

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[3] There is some confusion about the sign convention for scattering amplitudes. Most theorists use a convention wherein the scattering amplitude is negative for a repulsive potential and positive for a weak attractive potential. In experimental neutron physics, especially in neutron diffraction where these quantities are more used than anywhere else, the opposite convention is almost universal.

Fermi used the latter convention, but carefully called the quantity "scattering length" or "interaction constant." In this paper, we use the convention of the neutron-diffraction experimenters.

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[5] The resonances referred to here are the single-particle ones. There are of course many other resonances, which even occur at a few electron volts spacing for some nuclei. They usually are unimportant for scattering because they involve states in which the energy is shared among many nucleons, and is not easily concentrated on one nucleon which can then be reemitted. Their scattering widths are therefore small (<1 eV) and the scattering amplitude for thermal neutrons is very unlikely to be much affected by them. A significant exception is ^{113}Cd, which we have excluded from our analysis and from the experimental data.

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Generalized Finite Matrix Hamiltonians

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A Fermion representation is developed for truncated Bose operators. The eigenvalue analysis of finite matrix Hamiltonians becomes transparent by using this representation. Renormalization is discussed independent of of infinities for simple separable cases.

I. INTRODUCTION

Finite matrix Hamiltonians are operators whose Hilbert space matrix representations are finite dimensional. They have been introduced as pedagogic aids at widely disparate levels of sophistication ranging from elementary quantum mechanics[1] to interacting quantum field theory.[2] In this note the chasm between the various applications is bridged by introducing a simple representation for the finite matrix operators. Because these operators can be expressed in terms of the familiar anticommuting Fermi creators and annihilators, the ideas discussed here are within the grasp of the first year quantum mechanics student.

Originally,[1] finite matrix operators were obtained by Buchdahl by arbitrarily truncating the matrices for the Bose creators and annihilators at the Nth row and column. This procedure naturally modifies the commutation rules satisfied by the operators. Buchdahl then analyzed the standard harmonic oscillator Hamiltonian

$$H=\tfrac{1}{2}[P^2+Q^2]$$

using the modified commutation rule

$$[Q,P]=i[I-NK].$$

Here I is the unit operator, N is a positive integer, and K is an idempotent operator which commutes with H. The construction of the eigenvalues and eigenvectors for this Hamiltonian provides an interesting, though laborious, exercise in the ladder method.

In Sec. II the truncated harmonic oscillator is discussed as a specific example of the general