R. Golub

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1 Outline of the argument

reference: Demkov, YuN and Ostrovskii, VN
Zero Range Potentials and their applications in Physics
Plenum Press 1988

1.1 multiple scattering states

Assuming a system of point-like scatterers then wave function will have the form

$$\psi(r) = e^{i\overrightarrow{k}_o \cdot \overrightarrow{r}} + \sum_j c_j \frac{e^{ik\left|\overrightarrow{r}-\overrightarrow{R_j}\right|}}{\left|\overrightarrow{r}-\overrightarrow{R_j}\right|} \tag{1}$$

Now we wish $\psi(r)$ to satisfy the boundary conditions

$$\psi(r)_{\left|\overrightarrow{r}-\overrightarrow{R_i}\right|\to 0} = c_i \left| \frac{1}{\left|\overrightarrow{r}-\overrightarrow{R_i}\right|} - \alpha_i \right| \tag{2}$$

which is the boundary condition for a point scatterer with scattering amplitude $1/\alpha_i$. Combining these equations we have

$$c_i \left[\frac{1}{\left|\overrightarrow{r}-\overrightarrow{R_i}\right|} - \alpha_i \right] = e^{i\overrightarrow{k}_o \cdot \overrightarrow{R}_i} + c_i \left[\frac{1}{\left|\overrightarrow{r}-\overrightarrow{R}_I\right|} + ik \right] + \sum_{j\neq i} c_j \frac{e^{ikR_{ij}}}{R_{ij}} \tag{3}$$

where

$$R_{ij} = \left|\overrightarrow{R_i}-\overrightarrow{R_j}\right| \tag{4}$$

Thus

$$0 = e^{i\overrightarrow{k}_o \cdot \overrightarrow{R}_i} + c_i\left[\alpha_i + ik\right] + \sum_{j\neq i} c_j \frac{e^{ikR_{ij}}}{R_{ij}}$$

$$\begin{aligned} -e^{i\overrightarrow{k}_o \cdot \overrightarrow{R}_i} &= \left[\alpha_i + ik\right] c_i + \sum_{j\neq i} \frac{e^{ikR_{ij}}}{R_{ij}} c_j \\ &= \Gamma(k)_{ij}\, c_j \end{aligned}$$

where

$$\Gamma(k)_{ii} = \alpha_i + ik \tag{5}$$
$$\Gamma(k)_{i\neq j} = \frac{e^{ikR_{ij}}}{R_{ij}}$$

Thus

$$-c_j = \sum_i \left(\Gamma(k)^{-1}\right)_{ij} e^{i\overrightarrow{k}_o\cdot\overrightarrow{R}_i}$$

and

$$\psi(r) = e^{i\overrightarrow{k}_o\cdot\overrightarrow{r}} - \sum_{ji} \left(\Gamma(k)^{-1}\right)_{ij} e^{i\overrightarrow{k}_o\cdot\overrightarrow{R}_i} \frac{e^{ik\left|\overrightarrow{r}-\overrightarrow{R_j}\right|}}{\left|\overrightarrow{r}-\overrightarrow{R_j}\right|}$$
$$\psi(r\to\infty) \longrightarrow e^{i\overrightarrow{k}_o\cdot\overrightarrow{r}} - \frac{e^{ikr}}{r}\sum_{ji} \left(\Gamma(k)^{-1}\right)_{ij} e^{i\overrightarrow{k}_o\cdot\overrightarrow{R}_i} e^{-i\overrightarrow{k}\cdot\overrightarrow{R}_j}$$

So we see that the total scattering amplitude for the whole system of point particles is

$$F(k) = -\sum_{ji} \left(\Gamma(k)^{-1}\right)_{ij} e^{i\overrightarrow{k}_o\cdot\overrightarrow{R}_i} e^{-i\overrightarrow{k}\cdot\overrightarrow{R}_j}$$

and the forward scattering length is

$$\bar{a} = -F(0) = \sum_{ji} \left(\Gamma(0)^{-1}\right)_{ij} \tag{6}$$

1.1.1 Single scatterer

For a single scatterer we have

$$\Gamma(k)_{11} = \alpha_1 + ik$$

and

$$\bar{a} = \frac{1}{\alpha_1 + ik} = \frac{a_1}{1 + ika_1}$$

1.1.2 Two scattering centers - bound state

In this case we have to replace ik by $-\kappa$ and the condition for the system of equations (3) to have a solution is (note that for the bound states the term $e^{i\overrightarrow{k}_o\cdot\overrightarrow{R}_i}$ in (3) is absent):

$$\det\Gamma(i\kappa) = 0$$
$$\Gamma(i\kappa) = \begin{bmatrix} \alpha_1 - \kappa & \frac{e^{-\kappa R_{12}}}{R_{12}} \\ \frac{e^{-\kappa R_{12}}}{R_{12}} & \alpha_2 - \kappa \end{bmatrix} \tag{7}$$

Thus the energy of the bound state is determined by

$$(\alpha_1 - \kappa)(\alpha_2 - \kappa) = \frac{e^{-2\kappa R_{12}}}{R_{12}^2}$$

or taking $\alpha_1 = \alpha_2 = \alpha$ for simplicity

$$(\alpha - \kappa) = \pm \frac{e^{-\kappa R_{12}}}{R_{12}}$$

$$\kappa R - R/a = e^{-\kappa R}$$

which would indicate that $\kappa \to \infty$ as $R \to 0$, the energy of the bound state approaching $-\infty$. The scattering length for the 2 center system is given by equation (6) with

$$\Gamma(0) = \begin{bmatrix} \alpha & \frac{1}{R_{12}} \\ \frac{1}{R_{12}} & \alpha \end{bmatrix} \tag{8}$$

$$\Gamma(0)^{-1} = \begin{bmatrix} \frac{\alpha}{\alpha^2 R^2 - 1} R^2 & -\frac{R}{\alpha^2 R^2 - 1} \\ -\frac{R}{\alpha^2 R^2 - 1} & \frac{\alpha}{\alpha^2 R^2 - 1} R^2 \end{bmatrix} \tag{9}$$

so that

$$\bar{a} = \sum_{ji} \left(\Gamma(0)^{-1} \right) = \frac{2R(\alpha R - 1)}{\alpha^2 R^2 - 1} = \frac{2Ra}{(R + a)} \tag{10}$$

with $a = 1/\alpha$. This goes to zero as $R \to 0$ which is unphysical.

COMMENT: Both the binding energy and κ go to infinity as R goes to zero because the wave function is forced to adopt a stronger and stronger curvature to fit between the two centers (the two zero range potentials). Clearly in the limit when the two centers have collapsed into one there is no need for this strong curvature and it is seen that the limit is unphysical.