Chapter 2
Quantum Behavior

- Early 1900's - photoelectric effect. 
  Blackbody radiation
  quantized nature.
- first noticed that light had to be treated as both a wave and particle.

led to the deBroglie wavelength
\[ \lambda = \frac{h}{p} \]

not the whole story - this requires a fixed momentum - can't account for the dynamical behavior.

Could describe wave nature and particle nature, but could not reconcile the two.

Solution is to abandon the concept of particle size - size cannot be determined exactly - it's determined by the context of how you are measuring it. It's a coupling between the object and the observing system.

Example - want to measure where an electron is - we can confine it to be in some region. 
- the dimension \( \Delta x \) is determined by what experiment we do.
- could be dimension of an atom
- size of a block of material (conduction)
Uncertainty arguments hold for other measurements as well.

deBroglie relationship between the Energy $E$ and frequency

$$E = h\nu = k\lambda$$

So to measure $E$ precisely, one needs to measure for some length of time $t$.

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Implication: short-lived radioactive species can only determine $E$ based on $\Delta t$

i.e. $^{5}\text{He} \rightarrow {}^{4}\text{He} + n \quad \tau \sim 10^{-21}\text{s}$

Measure for entire life of particle $\Delta t \sim 10^{-21}\text{s}$

Can only know $E$ to an uncertainty of

$$\Delta E \left(10^{-21}\right) \geq 1.05 \times 10^{-34} \text{ m}^2 \text{kg/s}/\sqrt{2}$$

$$\Delta E \geq 5 \times 10^{-14} \tau = 0.32 \text{MeV}$$

This uncertainty in energy is called the energy width.

Only if system is stable, we can measure $E$ with arbitrary uncertainty.
So wave that characterizes the particle has a large amplitude in the region of \( \Delta x \) and small amplitude elsewhere.

Look at deBroglie relationship

\[ \lambda = \frac{h}{p_x} \]

This gives no information about position \( x \), so particle can be anywhere - no way to localize it. Thus the failure of the discrete relationships.

Turns out there is a tradeoff -

\[ \lambda = \frac{h}{p_x} \quad \text{meaning} \quad \Delta x \]

But no information about \( \Delta x \). In order to get information about \( \Delta x \), you need to trade off knowledge about \( p_x \).

If we make simultaneous measurements of \( p_x \) and \( x \), we can are bound by the Heisenberg Uncertainty relationship:

\[ \Delta x \Delta p_x \geq \frac{\hbar}{2} \]

We can measure \( \Delta x \) or \( \Delta p_x \) to arbitrary smaller precision, but not both!

- Same holds for \( y, z \) components.

We talk about a wave packet, which is a collection of waves represented by a range of \( p_x \) \( (\Delta p_x) \) and position \( x \) \( (\Delta x) \). What we colloquially denote as a particle is actually a wave packet!
angular momentum uncertainty

- classically, can determine $\Delta l_x, \Delta l_y, \Delta l_x$.

- quantum - improve our knowledge of one component at the expense of the other two.

\[ \Delta l_x \Delta \phi \geq \frac{h}{2} \]

So, if we imagine $l$ to be rotating about the $z$-axis, we can measure the phase ($\phi$) or projection to arbitrary accuracy, but not both.
So how do you work with the wave packets?

Schrödinger Equation:

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x)\]

Comes from \( E = \hbar \omega \) and \( \lambda = \frac{1}{p} \) (or \( p = \frac{h}{\lambda} = \hbar k \))

Schrödinger determined you can express phase of a plane wave using a complex phase:

\[ \psi(x) = \exp \left( i (kx - \omega t) \right) \]

He realized that

\[ \frac{\partial}{\partial t} \psi = -i \omega \exp \left( i (kx - \omega t) \right) = -i \omega \psi \]

\[ E \psi = \hbar \omega \psi = i \hbar \frac{\partial \psi}{\partial t} \]

\[ \frac{\partial}{\partial x} \psi = i k \exp \left( i (kx - \omega t) \right) = i k \psi \]

\[ p_x \psi = \hbar k \psi = -i \hbar \frac{\partial}{\partial x} \psi \]

\[ p_x^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} \psi \]
\[ \hat{p}^2 \psi = (\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) \psi = -\hbar^2 \left( \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right) \psi = -\hbar^2 \nabla^2 \psi \]

Using these relationships with the classical energy of a particle:

\[ E = \frac{\hat{p}^2}{2m} + V \]

\[ \text{Kinetic} \quad \text{Potential} \]

\[ \frac{i\hbar}{\sqrt{\hbar}} \frac{\delta}{\delta t} \psi = \frac{-\hbar^2}{2m} \nabla^2 \psi + V \psi \]

\[ \frac{-\hbar^2}{2m} \frac{\delta^2}{\delta x^2} \psi(x) + V(x) \psi(x) = E \psi(x) \]

Point out that for a free particle, \( V = 0 \).

Eq. will only have solutions for certain values of \( E \) - quantizations comes out for applying the boundary conditions.

General solution is \( \psi(x, t) = \psi(x) e^{-iwt} \) \( (\omega = \frac{E}{\hbar}) \)

Boundary condition \( \psi \) and \( \frac{\delta \psi}{\delta x} \) must be continuous across any boundary -

Same as for classical waves - i.e., optical -
Mathematically

\[ \lim_{\varepsilon \to 0} [4(a+\varepsilon) - 4(a-\varepsilon)] = 0 \]
\[ \lim_{\varepsilon \to 0} \left[ \left( \frac{df}{dx} \right)_{x=a+\varepsilon} - \left( \frac{df}{dx} \right)_{x=a-\varepsilon} \right] = 0 \]

Also - \( \Psi \) must remain finite.

Wave function allows one to actually calculate things.

Example - Probability of finding a particle between \( x \) and \( x + dx \)

\[ P(x) \, dx = \Psi^* (x, t) \Psi (x, t) \, dx \]

\[ \uparrow \]

Complex conjugate

\[ \Psi^* \Psi \equiv \text{probability density} \]

Prob. of finding particle between \( x_1 \) and \( x_2 \) is then

\[ P = \int_{x_1}^{x_2} \Psi^* \Psi \, dx \]

Note total probability of finding the particle must be 1.

\[ P = \int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1 \]
any function can be evaluated as well. think of it as an average value:

\[ f(x) \quad <f> = \int 4^* f(x) \, 4 \, dx \]

what does \(<f>\) mean?

If we make a large # of measurements of \(f(x)\), \(<f>\) will be the average. We can not predict what a single measurement will predict! All we can do is to predict the statistical distribution - one of the underlying foundation of Q.M.

what about quantities that are not a function \(f(x)\)?

ie - \(p_x\) ?

\[ p_x = -i \hbar \frac{\partial}{\partial x} \]

\[ <p_x> = \int 4^* (-i \hbar \frac{\partial}{\partial x}) \, 4 \, dx \]

note that operators only operate to the right.

\[ = -i \hbar \int 4^* \frac{\partial 4}{\partial x} \, dx \]

Also note time dependence - \(4(x,t) = 4(x)e^{-i\omega t}\)
\(4^*(x,t) = 4^*(x)e^{i\omega t}\)

time dependent part cancels - "Stationary State"
Particle current density:

\[ j = \frac{\hbar}{2mi} \left( \frac{\partial^2 \psi^*}{\partial x^2} - \kappa \frac{\partial \psi^*}{\partial x} \right) \]

# of particles per second passing through a point \( x \). Analogous to an electric current.

Already alluded to the 3-dimensional form:

\[-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi^*}{\partial x^2} + \frac{\partial^2 \psi^*}{\partial y^2} + \frac{\partial^2 \psi^*}{\partial z^2} \right) + V(x, y, z) \psi^*(x, y, z) = \mathcal{E} \psi^*(x, y, z) \]

\[ \psi(x, y, z, t) = \psi(x, y, z) e^{-i\omega t} \]

\( \psi^* \psi \) now probability per unit volume: \( \, dV = dx \, dy \, dz \)

\[ P = \int \psi^* \psi \, dV = 1 \]
Spherical Coordinates

\[-\frac{\hbar^2}{2m} \left[ \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] \]

\[+ V(r, \theta, \phi) \psi(r, \theta, \phi) = E \psi(r, \theta, \phi) \]

\[dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \]
Problems:

Free particle:

\[ V(x) = 0 \]

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi(x) \]

\[ \frac{d^2 \psi}{dx^2} = -\frac{2mE}{\hbar^2} \psi(x) \]

General solution:

\[ \psi(x) = A \sin(kx) + B \cos(kx) = A e^{ikx} + B e^{-ikx} \]

where: \[ k^2 = \frac{2mE}{\hbar^2} \]

Time-dependent solution:

\[ \psi(x, t) = A e^{i(kx - \omega t)} + B e^{-i(kx + \omega t)} \]

\[ \uparrow \quad \uparrow \]

Wave travelling \quad Wave travelling

\[ \rightarrow \quad \leftarrow \]

Intensities: \[ |A|^2, |B|^2 \]

No boundary conditions \Rightarrow all values of \( E \) are valid.

\[ \int \sin^2 \theta \text{ or } \cos^2 \theta \text{ don't converge - so can't normalize } \psi \]

Instead, apply physical condition -

\( \text{or} \) - Accelerator gives particles travelling one way or the other.

\[ \Rightarrow |A| \text{ or } |B| = 0 \]
Take $|\mathbf{B}| = 0$ \\
$\Rightarrow \psi = A e^{i(kx - \omega t)}$

Other quantity we know - current density - how many particles / sec. being emitted - $I$

\[
j = \frac{\hbar}{2mi} \left[ \psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right]
\]

\[
= \frac{1A^2\hbar}{2mi} \left[ e^{-i\chi} - e^{i\chi} \right] \left[ e^{i(kx - \omega t)} - e^{-i(kx - \omega t)} \right]
\]

\[
\Rightarrow \frac{1A^2\hbar}{2mi} (2i\chi) = \frac{1A^2\hbar k}{m} = I \quad \{ \text{keep - will need in step potential problem} \}
\]

\[
\Rightarrow 1A = \frac{mI}{\hbar k} \Rightarrow 1A = \sqrt{\frac{mI}{\hbar k}}
\]

\[
\psi = \sqrt{\frac{mI}{\hbar k}} e^{i(kx - \omega t)}
\]
Step Potential  \( E > E_0 \)

\[ V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases} \]

Region I  \( x = 0 \)  Region II

\[ E > V_0 \]

I. Same as free particle -

\[
\frac{d^2 \psi_1}{dx^2} = \frac{-2mE}{\hbar^2} \psi_1,
\]

\[ \psi_1 = A e^{ikx} + B e^{-ikx} \]

\[ k_1 = \sqrt{2mE}/\hbar \]

II. \(-\hbar^2/2m \frac{d^2 \psi_2}{dx^2} + V \psi_2 = E \psi_2 \)

\[
\frac{d^2 \psi_2}{dx^2} = \frac{-2m(E-V_0)}{\hbar^2} \psi_2
\]

\[ \psi_2 = C e^{ikx} + D e^{-ikx} \]

\[ k_2 = \sqrt{2m(E-V_0)}/\hbar \]

Apply boundary conditions -

at \( x = 0 \) \( \psi = \psi_2 \) \( \Rightarrow A + B = C + D \)

\[ \frac{d\psi_1}{dx} = \frac{d\psi_2}{dx} \Rightarrow ik_1A - ik_1B = ik_2C - ik_2D \]

\[ k_1(A - B) = k_2(C - D) \]
if wave starts at $x = -\infty$ (far left)

$A$: represents wave moving to right
$B$: reflected wave from barrier
$C$: wave transmitted through barrier
$D$: no physical correspondence $\Rightarrow D = 0$

Solve for $B, C$ in terms of $A$

\[
\begin{align*}
A + B &= C \\
A - B &= \frac{k_2}{k_1} C \\
2A &= (1 + \frac{k_2}{k_1}) C \\
C &= \frac{2A}{1 + \frac{k_2}{k_1}}
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
(\frac{k_2}{k_1} - 1)A + \left( \frac{k_2}{k_1} + 1 \right)B = 0 \\
B = \frac{(1 - \frac{k_2}{k_1}) A}{(1 + \frac{k_2}{k_1})}
\end{array} \right.
\]

Reflective coefficient $R = \frac{\text{reflected}}{\text{incident}}$
Transmission coeff $T = \frac{\text{transmitted}}{\text{incident}}$

Would expect $R + T = 1$

from free particle eq. $j = \frac{\hbar k_1}{m} |A|^2$ $\Rightarrow$ $j_{inc} = \frac{\hbar k_1}{m} |A|^2$

\[
R = \frac{j_{inc}}{j} = \frac{1|B|^2}{1|A|^2} = \left( \frac{1 - \frac{k_2}{k_1}}{1 + \frac{k_2}{k_1}} \right)^2
\]

\[
J_{trans} = \frac{\hbar}{2m} \left[ 4 \frac{dx}{dx} + 4 \frac{dx}{dx} \right] = (c_1 \frac{dx}{dx}) \frac{\hbar}{2m} (2k) = \frac{\hbar c^2}{2m} |c|^2
\]

\[
T = \frac{j_{trans}}{j_{inc}} = \frac{k_2 |c|^2}{k_1 |A|^2} = \frac{4k_2/k_1}{(1 + \frac{k_2}{k_1})^2}
\]
\[ R = \frac{1 - \frac{k_2}{k_1}}{(1 + \frac{k_2}{k_1})^2} \quad T = \frac{4k_2/k_1}{(1 + \frac{k_2}{k_1})^2} \]

\[ R + T = \frac{1 - 2 \frac{k_2}{k_1} + \left(\frac{k_2}{k_1}\right)^2 + 4k_2/k_1}{(1 + \frac{k_2}{k_1})^2} = \frac{1 + 2 \frac{k_2}{k_1} + \left(\frac{k_2}{k_1}\right)^2}{(1 + \frac{k_2}{k_1})^2} = 1 \]

**Core**

\[ E > E_0 \]

**STEP POTENTIAL**

**E < V_0**

**Region I** solution remains the same:

\[ \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2} \psi, \quad k_1 = \frac{\sqrt{2mE}}{\hbar} \]

\[ j_{inc} = (\hbar k_1/m) \text{Ai}^2 \]

**Region II:**

\[ -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0 \psi = E \psi \]

\[ \frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2} (E - V_0) \psi = \frac{2m}{\hbar^2} (V_0 - E) \psi \]

\[ \psi_2 = A e^{k_2 x} + B e^{-k_2 x} \quad k_2 = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \]

Exponential solution - not oscillatory
condition - wave function must be finite -

\[ \Rightarrow x \to \infty \Rightarrow e^{ikx} \to \infty \Rightarrow c = 0 \]

you have an incident wave (A), reflected wave (B), and transmitted wave (D) as before.

\[ \downarrow \text{E} \quad k=0 \quad \text{penetration into the barrier} \]

Classically forbidden region

\[ \Rightarrow \text{all particles reflected at boundary} \]

\[ \Rightarrow \text{wave packet can actually penetrate into barrier} \]

\[ \Rightarrow \text{actual particle can never penetrate into region} \]

\[ \text{ie - kinetic energy } < 0 \quad E < V_0 \]

\[ E = V_0 + K_S \]
Potential Barrier \( E > V_0 \)

\[ V = 0 \quad x < 0 \]
\[ V = V_0 \quad 0 \leq x < a \]
\[ V = 0 \quad x > a \]

\[ \psi = A e^{ik_1x} + B e^{-ik_1x} \]
\[ \psi = C e^{ik_2x} + D e^{-ik_2x} \]
\[ \psi = F e^{ik_3x} + G e^{-ik_3x} \]

\[ k_1 = k_3 = \sqrt{\frac{2mE}{\hbar^2}} \]
\[ k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \]

Take incident wave to be from \( x = -\infty \) \( G = 0 \)

Boundary Conditions \(- x = 0\)

\[ A + B = C + D \]
\[ C e^{ik_2a} + D e^{-ik_2a} = F e^{ik_1a} \]
\[ k_1A - k_3B = k_2C - k_2D \]
\[ k_2C e^{ik_2a} - k_2D e^{-ik_2a} = k_1F e^{ik_1a} \]

\[ T = \frac{|F|^2}{|A|^2} = \frac{1}{1 + \frac{1}{4} \frac{V_0^2}{E(E-V_0)} \sin^2 k_2a} \]

\( E < V_0 \)

\( \rightarrow \) same as above, but exponential solution now for middle part.

\( \rightarrow \) finite region so \( C \neq D \neq 0 \)

\[ T = \frac{1}{1 + \frac{1}{4} \frac{V_0^2}{E(V_0-E)} \sinh^2 k_2a} \]
E < V₀ case:
Classically, we would expect T = 0
(i.e., the ball doesn't have enough energy to get over the hill, so it tunnels through.)
called "barrier penetration" or Q.M. "tunneling"

E > V₀
Classically, we would expect the wave on "the other side of the hill" to be the same as the one on the left. We would not expect part of the ball to reflect...
Infinite Well

\[ V = \infty \quad V = 0 \quad V = \infty \]

\[ x = 0 \quad x = a \]

\[
\frac{-\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) \psi(x) = E \psi(x)
\]

Particle always between \( x = 0 \) and \( x = a \) - walls not penetrable.

\[ \psi = A e^{ikx} + B e^{-ikx} = A \sin(kx) + B \cos(kx) \]

Continuity condition (\( \psi(x+\varepsilon) = \psi(x-\varepsilon) \)) as \( \varepsilon \to 0 \) must hold.

\[ \psi(0) = 0 = A \sin(ka) + B \cos(ka) \]

\[ B = 0 \]

\[ \psi(a) = 0 = A \sin(ka) \]

\[ \sin(ka) = 0 \Rightarrow ka = n\pi \text{ integer} \]

\[ k = \frac{n\pi}{a} \Rightarrow E = \frac{n^2 \hbar^2}{2m} \]

\[ \psi(x) = \frac{1}{\sqrt{a}} \sin \left( \frac{n\pi x}{a} \right) \]

\[ E_n = \frac{n^2 \hbar^2}{2ma^2} \]

\[ A = \int_0^a \sin^2 \left( \frac{n\pi x}{a} \right) dx = 1 \]

\[ B = \frac{A^2}{\frac{1}{2} - \frac{1}{2}} = \frac{A^2}{\frac{1}{2}} \]

\[ \Delta A = \frac{\frac{1}{2}}{\frac{1}{2}} \]
INPLICATIONS -

1). Energy is quantized - only certain values permitted - known as "bound states"

\[ n = 1 \text{ ground state} \]

\[ n = 2 \text{ 1st excited state} \]

briefly introduce concept of energy levels - will discuss later -

\[ \text{Energy} \]

- \[ n = 4 \]
- \[ n = 3 \] "excited states"
- \[ n = 2 \]
- \[ n = 1 \text{ 1st excited state} \]

infinite # of solutions -
Finite Potential Well

I  \quad V(x) = V_0  \\
II \quad V(x) = 0  \\
III \quad V(x) = V_0

Take \( E < V_0 \) (inside the well)

\[ I \quad y = Ae^{k_1 x} + Be^{-k_1 x} \quad \text{exponentially decaying solution} \]
\[ II \quad y = Ce^{ik_2 x} + De^{-ik_2 x} \quad \text{as \& boundary condition} \]
\[ III \quad y = Fe^{k_1 x} + Ge^{-k_1 x} \]

\[ k_1 = \sqrt{2m(V_0 - E)/\hbar^2} \quad k_2 = \sqrt{2mE/\hbar^2} \]

Wave fn. finite \( \Rightarrow B = F = 0 \)

Wave fn. continuous \( \Rightarrow k = \frac{\pi}{2}, \frac{3\pi}{2} \)

1st derivative continuous \( \Rightarrow k = \frac{\pi}{2}, \pi = \frac{3\pi}{2} \)

\[ k_2 \tan \left( \frac{k_2 a}{2} \right) = k_1 \quad \text{must solve for numerically.} \]
\[ -k_2 \cot \left( \frac{k_2 a}{2} \right) = k_1 \]

\( \# \text{ of solutions determined by the well depth } V_0. \)

In nucleus, think of as \# of bound states.
Harmonic Oscillator

Look at some more general potentials — $V(x)$.

Expand $V(x)$ about $x_0$ — Taylor series

$$V(x) = V(x_0) + \left( \frac{dV}{dx} \right)_{x=x_0} (x-x_0) + \frac{1}{2} \left( \frac{d^2V}{dx^2} \right)_{x=x_0} (x-x_0)^2 + \ldots$$

$V(x)$ vs. $x$ graph

Let's choose $V(x) = \frac{1}{2} k x^2$ — harmonic oscillator.

Will not go through details of the solution.

$$E_n = \hbar \omega (n + \frac{1}{2}) \quad n=0, 1, 2, \ldots$$

$n=0$ $E = \frac{1}{2} \hbar \omega$

$n=1$ $E = \frac{3}{2} \hbar \omega$

$n=2$ $E = \frac{5}{2} \hbar \omega$

$n=3$ $E = \frac{7}{2} \hbar \omega$
3-dimensional well

\[ V(x, y, z) = 0 \quad 0 \leq x \leq a, \; 0 \leq y \leq a, \; 0 \leq z \leq a \]
\[ = \infty \; x > a, \; x < 0, \; y < 0, \; y > a, \; z < 0, \; z > a \]

"particle in a box"

outside box, \( \psi = 0 \)
inside box

\[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = E \psi(x, y, z) \]

solution \( \psi \):

\[ \psi(x, y, z) = \sqrt{\left( \frac{2}{a} \right)^3} \sin \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{a} \right) \sin \left( \frac{n_z \pi z}{a} \right) \]

\[ E_{n_x, n_y, n_z} = \frac{\hbar^2 \pi^2}{2ma^2} \left( n_x^2 + n_y^2 + n_z^2 \right) \]
\[ \begin{aligned} n_x &= 1, 2, 3, \ldots \\
    n_y &= 1, 2, 3, \ldots \\
    n_z &= 1, 2, 3, \ldots \\
\end{aligned} \]

There are independent

Ground state \( E(1, 1, 1) = \frac{3 \hbar^2 \pi^2}{2ma^2} \)

Wave function would be peaked at the center -
\( x = y = z = \frac{a}{2} \) and falling of to zero at the boundaries.
What about the first excited state?

\[ E(1,1,2) \quad E(1,2,1) \quad E(2,1,1) \]

each has the same energy. Wave functions for each are similar, but different. \( \rightarrow \) different probability distribution,

- known as "degeneracy"
- this example is 3-fold degenerate

\[
\begin{array}{c|c|c}
\text{Degeneracy} & \text{Energy} & E/E_0 \\
\hline
1 & \text{(2,1,1)} & 12 \\
3 & \text{(1,3,1) (3,1,1) (1,1,3)} & 11 \\
3 & \text{(1,2,1) (2,2,1) (2,1,2)} & 9 \\
3 & \text{(2,1,1) (1,2,1) (1,1,2)} & 6 \\
1 & \text{(2,2,2)} & 3 \\
\end{array}
\]

- not regularly spaced
- going to be important later on when talking about the shell model

\[ E_0 = \frac{\hbar^2 \pi^2}{2 ma^2} \]
Spherical well - "particle in a ball" from Schiff

- Use spherical coordinates -

\[ V(r, \theta, \phi) \] only depends on \( r \) in most cases, \( V(r) \)

\[ \psi = \psi(r, \theta, \phi) = R(r), \Theta(\theta), \Phi(\phi) \]

\[ = R(r) Y(\theta, \phi) \]

Assume spherically symmetric potential \( V(r) \)

\[ -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi + V(r) \psi = E \psi \]

\[ \psi(r, \theta, \phi) = R(r) Y(\theta, \phi) \]

\[ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{2m}{\hbar^2} \left( E - V(r) \right) = -\frac{1}{Y} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) \right. \]

\[ \left. + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right] \]

depends on \( r \)

depends on \( \theta, \phi \)

\[ \Rightarrow \] both sides must equal a constant - call \( \ell (\ell + 1) \)

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left\{ \frac{2m}{\hbar^2} \left[ E - V(r) \right] - \frac{\ell (\ell + 1)}{r^2} \right\} R = 0 \]

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} + \ell (\ell + 1) Y = 0 \]
separate answer piece \[ Y(\theta, \phi) = \Theta(\theta) \Phi(\phi) \]

follow same procedure:

\[ \frac{d^2 \Phi}{d \phi^2} + \frac{m_\phi^2}{\sin^2 \theta} \Phi = 0 \]

\[ m_\phi \text{ is the constant thin line} \]

\[ \frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d \Theta}{d \theta} \right) + \left( \ell (\ell + 1) - \frac{m_\theta^2}{\sin^2 \theta} \right) \Theta = 0 \]

Solution:

\[ m_\theta = 0 \Rightarrow \Theta = A + B \phi \]

\[ m_\theta \neq 0 \Rightarrow \Theta = A e^{i m_\theta \phi} + B e^{-i m_\theta \phi} \]

requirement that \[ \Psi \frac{d \Psi}{d \phi} \] be continuous (i.e. \( \Psi(0) = \Psi(2\pi) \))

leads to \( m_\phi \) being quantized as an integer:

\[ m_\phi = 0, \pm 1, \pm 2, \ldots \]

\[ \Psi = \frac{1}{2\pi} e^{i m_\phi \phi} \]

You can go through the same exercise with \( \Theta \), where the physically acceptable solutions are given by "Legendre polynomials";

\[ \Theta(\theta) = \left[ \frac{2\ell + 1}{2} \frac{(\ell - m_\phi)!}{(\ell + m_\phi)!} \right]^{1/2} P_{\ell}^{m_\phi}(\theta) \quad \ell = 0, 1, 2, \ldots \]

the \( Y(\theta, \phi) \) from before is actually the "spherical harmonics".

You should think of \( \Theta(\theta) \) and \( Y(\theta, \phi) \) like you did \( \sin \) and \( \cos \).
$P(\theta) \otimes \Psi(\theta, \phi)$ are orthogonal functions.

$L \otimes m$ are the quantization of angular momentum.

What about $R$?

Solution here is given in terms of "Bessel Functions" $j_l(kr)$ — again, orthogonal functions.

\[
\begin{array}{c|c|c|c|c|c}
3 & 2,1 & 1,3 & 4.0 & -
\
2 & 1,2 & 1,2 & 3.37 & -
\
1 & 1,0 & 1,0 & 2.05 & -
\
\end{array}
\]

- Not evenly spaced.
- Energy levels degenerate. Wave function with same $l$, different $m$ have same $E$.

\[\text{Degeneracy} = (2l+1) \text{ Energy} = \frac{E}{E_0}\]

See page 30 for probability distributions.
3-D harmonic oscillator:

\[ V(r) = \frac{1}{2}kr^2 \]

From before, the angular part yields \( y_n(\theta, \phi) \), a central potential.

\[ E_n = \hbar \omega_0 (n + \frac{3}{2}) \quad n = 0, 1, 2, \ldots \]

Some degeneracy in \( l \), plus degeneracy in \( m \) \((2l+1)\)

\[ l \leq n \quad \text{never} \rightarrow \text{never} \]

\[ n \leq l \rightarrow \text{label} \]

mention \( \frac{1}{2} \hbar \omega_0 \) per degree of freedom.

Coulomb potential:

\[ -V(r) = -\frac{Ze^2}{4\pi \varepsilon_0 r} \quad - \text{again a central potential.} \]

\[ E_n = -\frac{mZ^2e^4}{32\pi^2\varepsilon_0^2\hbar^2n^2} \quad l = 0, 1, \ldots, (n-1) \]

\[ n = 1, 2, 3, \ldots \]

Hydrogen atom

\[ E_n = -13.6 \text{eV}/n^2 \]
3-dimensional models:

- Energy levels are degenerate
- For central potentials, the wave function can be assigned a definite quantum number - \( l \) angular momentum.

Quantized angular momentum

For central potentials, \( \ell \) has the same functional form known as the "angular momentum quantum number."

Classically - \( \vec{L} = \vec{r} \times \vec{p} \)

Find magnitude - \( \langle \ell^2 \rangle \)

\[
\begin{align*}
P_x &= -i \hbar \frac{\partial}{\partial x} \quad P_y &= \ldots \\
&\qquad \implies L_x = yP_y - zP_z \\
\end{align*}
\]

\[
L^2 = L_x^2 + L_y^2 + L_z^2 \quad \implies \langle \ell^2 \rangle = \hbar^2 \ell (\ell + 1)
\]

\( \ell \) angular momentum is a constant of the motion (as in classical mechanics)
have length, what about its components? $l_x, l_y, l_z$.

Remember $\Delta l \Delta \Phi \geq \hbar/2$

So, if we measure $l_z$, we make $l_x, l_y$ indeterminate.
and vice versa.

Or - if measure $l_x$, then measure $l_y$, measuring $l_y$
destroys all knowledge we have of $l_z$.

By convention, we usually measure $l_z$.

$$\langle l_z \rangle = \hbar m, \quad m_z = 0, \pm 1, \pm 2, \ldots, \pm l$$

$$|\langle l_z \rangle| < 1, |l_z| = \hbar \sqrt{l(l+1)}$$

$z$-component always less than total length as you work.

Expect,

conventional Vector notation
Look at an atom's electronic states:

l's correspond to different electronic substates

May have heard them referred to as

s-state, p-states, etc.

l 0 1 2 3 4 5 6

s p d f g h i

We will use the same nomenclature when talking about nuclear states.

So we have the orbital angular momentum. There's actually a second component called an "intrinsic angular momentum" or a "spin". This does not have a classical analog. Picture of a spinning top is not correct.

\[
\langle s^2 \rangle = \hbar^2 s(s+1)
\]
\[
\langle s_z \rangle = \hbar m_s \quad m_s = \pm \frac{1}{2}
\]

Vector \( \vec{s} \) with possible \( z \)-components \( \pm \hbar/2 \)

Total angular momentum is

\[
\vec{j} = \vec{l} + \vec{s}
\]
\[
\langle j^2 \rangle = \hbar^2 j(j+1)
\]
\[
\langle j_z \rangle = \langle l_z + s_z \rangle = \hbar m_j
\]
\[
m_j = m_l + m_s = m_l \pm \frac{1}{2}
\]
\[ j = \frac{l + s}{2} = \frac{l}{2} + \frac{s}{2} \]  
\[ \text{integ} \Rightarrow j = \frac{1}{2}, j = \frac{3}{2} \]

\[ j \]-value usually included in notation

\[ l = 1 \text{ (p-state)} \Rightarrow j = \frac{3}{2}, \frac{1}{2} \Rightarrow j = \frac{1}{2}, j = \frac{3}{2} \]

\[ \frac{1}{2}, \frac{3}{2} \text{ states} \]

**Conservation Laws** - expect none of the observable properties to change as a result of the transformation.

**Parity**

\[ \hat{P} \rightarrow -\hat{P} \]

\[ V(\vec{r}) = V(-\vec{r}) \Rightarrow |\psi(\vec{r})|^2 = |\psi(-\vec{r})|^2 \]

**Charge Conjugation**

replace all particles with antiparticles

**Time Reversal Violation**

\[ t \rightarrow -t \]

\[ |\psi(t)|^2 = |\psi(-t)|^2 \]
Concentrate on parity for now:

\[ |\psi(\mathbf{r})|^2 = |\psi(-\mathbf{r})|^2 \Rightarrow \psi(\mathbf{r}) = \pm \psi(-\mathbf{r}) \]

\[ \psi(\mathbf{r}) = \psi(-\mathbf{r}) \Rightarrow \text{even parity} \]

\[ \psi(-\mathbf{r}) = -\psi(\mathbf{r}) \Rightarrow \text{odd parity} \]

Mixed parity wave functions do not occur.

For central potentials -

\[ Y_{lm}(\theta, \phi) \Rightarrow Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l Y_{lm}(\theta, \phi) \]

If \( V(\mathbf{r}) \) depends only on \( r^2 \) (i.e., harmonic oscillator), then \( l \) determines the parity of the wave function.

\( l \)-even \( \Rightarrow \) even parity

\( l \)-odd \( \Rightarrow \) odd parity

If

\[ |\psi(\mathbf{r})|^2 \neq |\psi(-\mathbf{r})|^2 \Rightarrow V(\mathbf{r}) \neq V(-\mathbf{r}) \]

\( \therefore \) System violates parity

Only experimentally observed in \( \beta \)-decay (weak interaction)

'57 - Ambler/Wu exp. 60Co.
Quantum Statistics:
revolves around indistinguishability of particles.

Take 2 electrons for example.

If the electrons are truly indistinguishable, then the probability density is symmetric with exchange of 2 particles.
\[ \psi_{12} = \pm \psi_{12} \]
\[ \Rightarrow \text{symmetric wave function} \]
\[ \Rightarrow \text{anti-symmetric wave function.} \]

Again - can't have mixed wave function (both symmetric and anti-symmetric)
as a result - particles with
integer spins \( \Rightarrow \text{symmetric} \)
\( \frac{1}{2} \)-integer spins \( \Rightarrow \text{anti-symmetric} \).

\[ \psi_{12} = \frac{1}{\sqrt{2}} \left[ \psi_A \left( \vec{r}_1 \right) \psi_B \left( \vec{r}_2 \right) \pm \psi_B \left( \vec{r}_1 \right) \psi_A \left( \vec{r}_2 \right) \right] \]
\[ + \text{ symmetric } - \text{ anti-symmetric.} \]

\( \Rightarrow \) can't have 2 identical particles of \( \frac{1}{2} \)-integer spin
in the same quantum state - "Pauli Exclusion Principle."